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J. Differential Equations 245 (2008) 80–101

Journal of Differential Equations

www.elsevier.com/locate/jde

Analytic invariant manifolds for sequences of diffeomorphisms ☆

Luis Barreira *, Claudia Valls

Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

Received 29 March 2007

Abstract

We obtain real *analytic* invariant manifolds for trajectories of maps assuming only the existence of a *nonuniform* exponential behavior. We also consider the more general case of sequences of maps, which corresponds to a nonautonomous dynamics with discrete time. We emphasize that the maps that we consider are defined in a real Euclidean space, and thus, one is not able to obtain the invariant manifolds from a corresponding procedure to that in the nonuniform hyperbolicity theory in the context of holomorphic dynamics. We establish the existence both of stable (and unstable) manifolds and of center manifolds. As a byproduct of our approach we obtain an exponential control not only for the trajectories on the invariant manifolds, but also for all their derivatives.

MSC: 37D10; 37D25

Keywords: Analytic invariant manifolds; Nonuniform hyperbolicity

1. Introduction

Our main objective is to establish the existence of *analytic* invariant manifolds for trajectories of real analytic maps satisfying the weakest possible hyperbolic behavior. Namely, we only assume that the trajectories admit a *nonuniform* exponential behavior. We still require some amount of partial hyperbolicity to establish the existence of the invariant manifolds, but this

☆ Supported by the Center for Mathematical Analysis, Geometry, and Dynamical Systems, and through Fundação para a Ciência e a Tecnologia by Program POCTI/FEDER, Program POSI, and the grant SFRH/BPD/26465/2006.

* Corresponding author.

E-mail addresses: barreira@math.ist.utl.pt (L. Barreira), cvalls@math.ist.utl.pt (C. Valls).

0022-0396/$ – see front matter

doi:10.1016/j.jde.2008.03.025

hyperbolicity can be spoiled exponentially along each trajectory as the initial time changes. In particular, in the case of nonuniform exponential dichotomies we obtain stable manifolds (and unstable manifolds), and in the case of nonuniform exponential trichotomies we obtain center manifolds. We also consider the more general case of sequences of maps, which corresponds to a nonautonomous dynamics with discrete time.

We now give a more detailed description of our results. We first briefly describe the setup. For each $m \in \mathbb{Z}$, let A_m be an invertible $n \times n$ real matrix, and let $f_m : \mathbb{R}^n \to \mathbb{R}^n$ be an analytic map with $f_m(0) = 0$. We consider the trajectories $(v_m)_{m \in \mathbb{Z}} \subset \mathbb{R}^n$ satisfying

$$v_{m+1} = A_m v_m + f_m(v_m) \tag{1}$$

for every $m \in \mathbb{Z}$. We notice that the constant sequence $v_m \equiv 0$ is a trajectory of (1). For simplicity, we assume in addition that the matrices A_m have the block form

$$A_m = \begin{pmatrix} B_m & 0 \\ 0 & C_m \end{pmatrix} \quad \text{for every } m \in \mathbb{Z}, \tag{2}$$

with an appropriate exponential behavior for each of the blocks, as detailed below. We note that the theory is already quite involved even when the matrices A_m take the form in (2). We obtain two main types of results:

1. When the blocks B_m and C_m correspond respectively to the stable and center-unstable components of a *weak nonuniform exponential dichotomy* (see Section 2.1 for the definition), we show that if each perturbation f_m has a holomorphic extension to the unit polydisk Δ with Lipschitz constant

$$\mathrm{Lip}(f_m|\Delta) \leqslant \delta e^{-\delta|m|} \quad \text{for every } m \in \mathbb{N} \tag{3}$$

   and some $\delta > 0$ sufficiently small, then there is a sequence $\mathcal{V}_m$, $m \in \mathbb{N}$, of analytic invariant stable manifolds for the trajectory $v_m \equiv 0$ that are tangent at zero to the real Euclidean spaces corresponding to the stable blocks B_m. This is the content of Section 2.
2. When the blocks B_m and C_m correspond respectively to the center and stable-unstable components of a *weak nonuniform exponential trichotomy with isometric central part* (see Section 5.1 for the definition), we show that if each perturbation f_m has a holomorphic extension to the unit polydisk Δ without central component and with Lipschitz constant satisfying (3), for every $m \in \mathbb{Z}$ and some $\delta > 0$ sufficiently small, then there is a sequence $\mathcal{V}_m$, $m \in \mathbb{Z}$, of analytic invariant center manifolds for the trajectory $v_m \equiv 0$ that are tangent at zero to the real Euclidean spaces corresponding to the central blocks B_m. This is the content of Section 5.

We can also obtain unstable manifolds, simply by reversing time in the case of stable manifolds. We emphasize that when we want to establish the existence of stable manifolds (or unstable manifolds) one can simply consider one-sided sequences $(v_n)_{n \in \mathbb{N}}$.

Our work naturally belongs to the theory of nonuniformly hyperbolic dynamics. In a certain sense, this is indeed the weakest possible setting in which one can construct the desired invariant manifolds. See [1] for a detailed exposition of the theory. The classical notions of exponential dichotomy and exponential trichotomy demand considerably from the dynamics and it is of interest to look for more general types of hyperbolic behavior, that can be much more typical. This is precisely what happens with the notions of nonuniform exponential dichotomy and nonuniform exponential trichotomy. Invariant *stable* and *unstable* manifolds were first obtained for nonuniformly hyperbolic trajectories by Pesin in his landmark work [7]. See [1] for details and further references. On the other hand, to the best of our knowledge, the first *center manifold theorem* in the nonuniformly hyperbolic setting was obtained in our recent work [2].

In the case of nonuniformly hyperbolic holomorphic dynamics in $\mathbb{C}^n$, the existence of invariant stable and unstable complex manifolds was announced by Wu in [8], unfortunately without giving a proof, and referring instead to his doctoral thesis. The proof is a combination of Pesin's results with the Cauchy–Riemann equations and the complex structure of $\mathbb{C}^n$. Furthermore, in the case of polynomial automorphisms of $\mathbb{C}^2$ it was shown independently by Wu in [8] and Bedford, Lyubich and Smillie in [3] (both developing the approach of Pesin in the "classical" nonuniform hyperbolicity theory) that with respect to the unique measure μ of maximal entropy, the stable and unstable manifolds of almost every point are conformally equivalent to the complex plane. The polynomial automorphisms of $\mathbb{C}^2$ can arguably be considered the simplest interesting examples of invertible holomorphic dynamics. The measure μ was introduced by Sibony (see [3,4]) and it was shown to be the unique measure of maximal entropy in [3]. The more general case of arbitrary holomorphic diffeomorphisms of a complex manifold was considered by Jonsson and Varolin in [6], where they showed that for each Lyapunov regular trajectory the stable and unstable manifolds are complex manifolds biholomorphic to a complex Euclidean space.

Here we consider instead the case of *real analytic* dynamics. To the best of our knowledge, it exists nowhere in the literature a corresponding stable manifold theorem in the nonuniformly hyperbolic setting that covers this situation. Furthermore, there exists also no corresponding center manifold theorem in the nonuniformly hyperbolic setting, even for *holomorphic* dynamics (i.e., for complex analytic dynamics). One could try to establish the existence of invariant manifolds using the results described above for holomorphic dynamics, but the fact that a given real analytic map may have singularities arbitrarily close to the real Euclidean space in which it is defined prevents us to proceed in this manner, at least without further hypotheses and modifications. We chose not to follow this path and to proceed instead in a natural manner from the point of view of real analytic dynamics.

2. Stable manifolds

In this section we study the existence of invariant stable manifolds composed of trajectories $(v_m)_{m\in\mathbb{N}}$ of (1). More precisely, when the matrices A_m possess stable and center-unstable components B_m and C_m (see (2)), we establish the existence of *analytic* invariant stable manifolds, with appropriate hypotheses on the analytic perturbations f_m in (1) (see Theorem 1 below). We can obtain in a straightforward manner an analogous statement concerning the existence of invariant unstable manifolds, and thus we omit the details.

2.1. Setup

We assume that:

(S1) there exist invertible $k \times k$ real matrices A_m, $m \in \mathbb{N}$, such that for some invariant decomposition $\mathbb{R}^k = E \times F$ (independent of m) we have

$$A_m = \begin{pmatrix} B_m & 0 \\ 0 & C_m \end{pmatrix} \quad \text{for each } m \in \mathbb{N}. \tag{4}$$

Due to the block form in (4), each sequence $(v_m)_{m\in\mathbb{N}} \subset \mathbb{R}^k$ satisfying $v_{m+1} = A_m v_m$ for every $m \in \mathbb{N}$ can be written in the form

$$v_m = \big(\mathcal{B}(m,n)x_n, \mathcal{C}(m,n)y_n\big) \quad \text{for every } m \geqslant n \geqslant 0,$$

where $v_n = (x_n, y_n) \in E \times F$, and for each $m \geqslant n \geqslant 0$,

$$\mathcal{B}(m,n) = \begin{cases} B_{m-1}\cdots B_n, & m > n, \\ \mathrm{Id}, & m = n, \end{cases} \qquad \mathcal{C}(m,n) = \begin{cases} C_{m-1}\cdots C_n, & m > n, \\ \mathrm{Id}, & m = n. \end{cases}$$

We say that the sequence $(A_m)_{m\in\mathbb{N}}$ admits *a weak nonuniform exponential dichotomy* if there exist constants $a > 0$, $b \geqslant 0$, $\varepsilon \geqslant 0$, and $D \geqslant 1$ such that for every $m \geqslant n \geqslant 0$ we have

$$\big\|\mathcal{B}(m,n)\big\| \leqslant De^{-a(m-n)+\varepsilon n}, \qquad \big\|\mathcal{C}(m,n)^{-1}\big\| \leqslant De^{-b(m-n)+\varepsilon m}. \tag{5}$$

One can think of the constants a and b as Lyapunov exponents, while the nonuniformity of the exponential behavior is controlled by the constant ε.

We now consider the l^∞ norm in $\mathbb{R}^k$, and we denote by $B(\mathbb{R}^k) \subset \mathbb{R}^k$ the unit ball centered at 0. We also consider the space $\mathcal{H}$ of analytic functions $f : B(\mathbb{R}^k) \to \mathbb{R}^k$ with $f(0) = 0$ and $d_0 f = 0$, admitting a holomorphic extension $\tilde{f}$ to the interior of the polydisk

$$\Delta\big(\mathbb{R}^k\big) = \big\{(z_1, \dots, z_k) \in \mathbb{C}^k\colon\ |z_i| \leqslant 1 \text{ for } i = 1, \dots, k\big\} \tag{6}$$

which is continuous on $\Delta(\mathbb{R}^k)$. We equip the space $\mathcal{H}$ with the norm

$$\|f\| := \sup\left\{\frac{\|\tilde{f}(u) - \tilde{f}(v)\|}{\|u - v\|}\colon\ u, v \in \Delta\big(\mathbb{R}^k\big) \text{ with } u \neq v\right\}.$$

We assume that:

(S2) there exist maps $f_m \in \mathcal{H}$, $m \in \mathbb{N}$, and a constant $\delta \in (0,1)$ such that

$$\|f_m\| \leqslant \delta e^{-2\varepsilon m} \quad \text{for each } m \in \mathbb{N}. \tag{7}$$

2.2. Lyapunov norms

Because of the nonuniform exponential behavior of the norm bounds in (5), we now introduce Lyapunov norms as in the nonuniform hyperbolicity theory (see for example [1]). We denote by $\widetilde{E} = E \otimes_{\mathbb{R}} \mathbb{C} \subset \mathbb{C}^k$ the complexification of the vector space $E \subset \mathbb{R}^k$. We fix $\varrho > 0$ such that $a - 2\varrho > 0$, and for each $m \in \mathbb{N}$ we define

$$\|u\|'_m = \sum_{k=m}^{+\infty} \|\mathcal{B}(k,m)u\| e^{(a-\varrho)(k-m)} \quad \text{for } u \in \widetilde{E},$$
$$\|v\|'_m = \sum_{k=0}^{m} \|\mathcal{C}(m,k)^{-1}v\| e^{(b-\varrho)(m-k)} \quad \text{for } v \in \widetilde{F}. \tag{8}$$

Using (5) it is straightforward to verify that the series in (8) converges, and setting $C = D/(1-e^{-\varrho})$, for each $u \in \widetilde{E}$, $v \in \widetilde{F}$, and $m \in \mathbb{N}$ we have

$$\|u\| \leqslant \|u\|'_m \leqslant Ce^{\varepsilon m}\|u\|, \qquad \|v\| \leqslant \|v\|'_m \leqslant Ce^{\varepsilon m}\|v\|. \tag{9}$$

We also set

$$B_n(E) = \{x \in E\colon \|x\|'_n \leqslant 1\}, \qquad \Delta_n(E) = \{z \in \widetilde{E}\colon \|z\|'_n \leqslant 1\}. \tag{10}$$

2.3. Existence of stable manifolds

Let now $\mathcal{X}$ be the space of sequences $(\varphi_m)_{m\in\mathbb{N}}$ of analytic functions $\varphi_m : B_m(E) \to F$, $m \in \mathbb{N}$, with a holomorphic extension $\tilde{\varphi}_m$ to the interior of $\Delta_m(E)$ (see (10)) which is continuous on $\Delta_m(E)$, and such that for every $m \in \mathbb{N}$ we have $\varphi_m(0) = 0$, $d_0\varphi_m = 0$, and

$$\|\varphi_m\| := \sup\left\{\frac{\|\tilde{\varphi}_m(\xi) - \tilde{\varphi}_m(\bar{\xi})\|}{\|\xi - \bar{\xi}\|}\colon \xi, \bar{\xi} \in \Delta_m(E) \text{ with } \xi \neq \bar{\xi}\right\} \leqslant 1. \tag{11}$$

Setting $\bar{\xi} = 0$ in (11), we obtain $\|\tilde{\varphi}_m(\xi)\| \leqslant \|\xi\|$ and hence

$$\varphi_m(B_m(E)) \subset B(F) \quad \text{and} \quad \tilde{\varphi}_m(\Delta_m(E)) \subset \Delta(F). \tag{12}$$

Thus

$$(\xi, \tilde{\varphi}_m(\xi)) \in \Delta(\mathbb{R}^k) \quad \text{for every } \xi \in \Delta_m(E) \subset \Delta(E) \subset \widetilde{E}.$$

In view of the assumption (S2) we can compute $\tilde{f}_m(\xi, \tilde{\varphi}_m(\xi))$ for $\xi \in \Delta_m(E)$.

Given a sequence $\varphi = (\varphi_m)_m \in \mathcal{X}$, for each $m \in \mathbb{N}$ we consider the graph

$$\mathcal{V}_m = \{(\xi, \varphi_m(\xi))\colon \xi \in B_m(E)\} \subset \mathbb{R}^k. \tag{13}$$

We also consider the maps $F_m = A_m + f_m$, and given $m \geqslant n \geqslant 0$ and $\xi \in B_n(E)$ we set

$$v_{mn}(\xi) = \mathcal{F}(m,n)(\xi, \varphi_n(\xi)), \tag{14}$$

where

$$\mathcal{F}(m,n) = \begin{cases} F_{m-1} \circ \cdots \circ F_n, & m > n, \\ \mathrm{Id}, & m = n. \end{cases}$$

We now present our stable manifold theorem.

Theorem 1. *Assume that* (S1)–(S2) *hold. If* $(A_m)_{m\in\mathbb{N}}$ *admits a weak nonuniform exponential dichotomy, then provided that* δ *in* (7) *is sufficiently small there exists a unique* $\varphi \in \mathcal{X}$ *such that*

$$\mathcal{F}(m,n)(\mathcal{V}_n) \subset \mathcal{V}_m \quad \text{for every } m \geqslant n \geqslant 0. \tag{15}$$

In addition:

1. $\mathcal{V}_m$ *is an analytic manifold,* $0 \in \mathcal{V}_m$, *and* $T_0\mathcal{V}_m = E$ *for every* $m \in \mathbb{N}$;
2. *for every* $\varrho \in (0, a/2)$ *there exists* $K > 0$ *such that given* $m \geqslant n \geqslant 0$ *and* $\xi, \bar{\xi} \in B_n(E)$ *we have*

$$\left\| v_{mn}(\xi) - v_{mn}(\bar{\xi}) \right\| \leqslant K e^{(-a+2\varrho)(m-n)+\varepsilon n} \|\xi - \bar{\xi}\|. \tag{16}$$

The proof of Theorem 1 is given in Section 3. We call each manifold $\mathcal{V}_m$ in Theorem 1 a *local stable manifold*. In Section 4 (see Theorem 2) we obtain an exponential decay analogous to that in (16) for the derivatives of v_{mn} along the stable manifold.

The fact that the definition of $\mathcal{V}_m$ in (13) involves the ball $B_m(E)$, which may depend on m unlike the ball $B(E)$, is a manifestation of the nonuniform exponential behavior in (5). Note that $B_m(E) \subset B(E)$ for every $m \in \mathbb{N}$ (see (9)), and that the size of $B_m(E)$ with respect to a fixed norm in E may decrease exponentially with m (again by (9)), at most with speed ε. This means that the sizes of the stable manifolds $\mathcal{V}_m$ may decrease exponentially with m along the zero trajectory of (1), although this will happen at smaller speed than the speed of trajectories on the stable manifolds (as given by (16)), provided that $a - 2\varrho > \varepsilon$. When $\varepsilon = 0$ the sizes of the stable manifolds $\mathcal{V}_m$ are uniformly bounded from below along the trajectory.

3. Proof of Theorem 1

The proof will be given in several steps. We first introduce some auxiliary function spaces, and we establish some preliminary results.

3.1. Function spaces

We write $f_m = (g_m, h_m) \in E \times F$ for each $m \in \mathbb{N}$. In view of the invariance property in (15), each trajectory starting in a given graph $\mathcal{V}_n$ must be in $\mathcal{V}_m$ for every $m \geqslant n$. Thus, given $n \in \mathbb{N}$ and $\xi \in B_n(E)$, and setting $v_n = (\xi, \varphi_n(\xi)) \in \mathcal{V}_n$ we must have

$$\mathcal{F}(m,n)v_n = \big(x_m(\xi), \varphi_m\big(x_m(\xi)\big)\big)$$

for some $x_m(\xi) \in E$, and each equation in (1) can be written in the form

$$x_m(\xi) = \mathcal{B}(m,n)\xi + \sum_{l=n}^{m-1} \mathcal{B}(m,l+1) g_l\big(x_l(\xi), \varphi_l\big(x_l(\xi)\big)\big),$$

$$\varphi_m\big(x_m(\xi)\big) = \mathcal{C}(m,n)\varphi_n(\xi) + \sum_{l=n}^{m-1} \mathcal{C}(m,l+1) h_l\big(x_l(\xi), \varphi_l\big(x_l(\xi)\big)\big) \tag{17}$$

for each $m \geqslant n$. The norm in $\mathbb{R}^k$ is given by $\|(x, y)\| = \max\{\|x\|, \|y\|\}$ for each $(x, y) \in E \times F$, and we equip the space $\mathcal{X}$ with the norm

$$|||\varphi||| = \sup\bigl\{\bigl\|\tilde{\varphi}_m(x)\bigr\|/\|x\|\colon\ m \in \mathbb{N} \text{ and } x \in \Delta_m(E) \setminus \{0\}\bigr\}. \tag{18}$$

One can easily verify that $\mathcal{X}$ is a complete metric space with this norm.

For a fixed $n \in \mathbb{N}$, given $m \geqslant n$ we set

$$\rho(m) = (-a + 2\varrho)(m - n). \tag{19}$$

Furthermore, we let $\mathcal{B}_n$ be the space of sequences $(x_m)_{m\geqslant n}$ of analytic functions x_m: $B_n(E) \to E$, for each $m \geqslant n$, admitting a holomorphic extension $\tilde{x}_m$ to the interior of $\Delta_n(E)$ which is continuous on $\Delta_n(E)$, such that $x_n(\xi) = \xi$ for each $\xi \in B_n(E)$, and

$$\|x\|' := \sup\bigl\{\bigl\|\tilde{x}_m(\xi)\bigr\|_m' e^{-\rho(m)}/\|\xi\|_n'\colon\ m \geqslant n \text{ and } \xi \in \Delta_n(E) \setminus \{0\}\bigr\} \leqslant 1. \tag{20}$$

It follows from (20) and (19) that

$$x_m\bigl(B_n(E)\bigr) \subset B_m(E) \subset B(E), \qquad \tilde{x}_m\bigl(\Delta_n(E)\bigr) \subset \Delta_m(E) \subset \Delta(E). \tag{21}$$

One can verify in a straightforward manner that $\mathcal{B}_n$ is complete metric space with the norm in (20).

3.2. *Solution on the stable direction*

We now establish the existence of a unique sequence $x = (x_m)_{m\geqslant n} \in \mathcal{B}_n$ satisfying the first equation in (17) for a given $\varphi \in \mathcal{X}$.

Lemma 1. *Provided that δ is sufficiently small, for each $\varphi \in \mathcal{X}$ and $n \in \mathbb{N}$ there exists a unique $x_\varphi = (x_{m,\varphi})_{m\geqslant n} \in \mathcal{B}_n$ satisfying the first equation in* (17) *for every $m \geqslant n$.*

Proof. Given $m \geqslant n$ and $\xi \in B_n(E)$, we define for each $x \in \mathcal{B}_n$ the operator

$$(Jx)_m(\xi) = \mathcal{B}(m, n)\xi + \sum_{l=n}^{m-1} \mathcal{B}(m, l + 1)g_l\bigl(x_l(\xi), \varphi_l\bigl(x_l(\xi)\bigr)\bigr).$$

When $\xi \in \Delta_n(E)$ we have $(\tilde{x}_l(\xi), \tilde{\varphi}_l(\tilde{x}_l(\xi))) \in \Delta(E)$ (see (12) and (21)), and thus $(Jx)_m$ admits a holomorphic extension to the interior of $\Delta_n(E)$, which we continue to denote $(Jx)_m$, given by

$$(Jx)_m(\xi) = \mathcal{B}(m, n)\xi + \sum_{l=n}^{m-1} \mathcal{B}(m, l + 1)\tilde{g}_l\bigl(\tilde{x}_l(\xi), \tilde{\varphi}_l\bigl(\tilde{x}_l(\xi)\bigr)\bigr).$$

Furthermore, $(Jx)_m$ is continuous on $\Delta_n(E)$, and clearly $(Jx)_n(\xi) = \xi$.

We now show that $\|(Jx)_m\|_m' \leqslant 1$ for each $m \geqslant n + 1$ (when $m = n$ this is immediate from the definitions). Setting

$$g_l^*(\xi) := \tilde{g}_l\bigl(\tilde{x}_l(\xi), \tilde{\varphi}_l\bigl(\tilde{x}_l(\xi)\bigr)\bigr), \tag{22}$$

and using (7), (11), (9), and (20), we obtain

$$\begin{aligned}\big\|g_l^*(\xi)\big\| &\leqslant \delta e^{-2\varepsilon l}\max\big\{\big\|\tilde{x}_l(\xi)\big\|, \big\|\tilde{\varphi}\big(\tilde{x}_l(\xi)\big)\big\|\big\}\\ &\leqslant \delta e^{-2\varepsilon l}\big\|\tilde{x}_l(\xi)\big\| \leqslant \delta e^{-2\varepsilon l}\big\|\tilde{x}_l(\xi)\big\|_l' \leqslant \delta e^{-2\varepsilon l}e^{\rho(l)}\|\xi\|_n'.\end{aligned} \tag{23}$$

By (8), (5), and since

$$\mathcal{B}(k,m)\mathcal{B}(m,n)=\mathcal{B}(k,n)$$

for each $k \geqslant m \geqslant n+1$, we obtain

$$\begin{aligned}\big\|(Jx)_m(\xi)\big\|_m' &= \sum_{k\geqslant m}\big\|\mathcal{B}(k,m)(Jx)_m\big\|e^{(a-\varrho)(k-m)}\\ &\leqslant \sum_{k\geqslant m}\big\|\mathcal{B}(k,m)\mathcal{B}(m,n)\xi\big\|e^{(a-\varrho)(k-m)}\\ &\quad + \sum_{k\geqslant m}\sum_{l=n}^{m-1}\big\|\mathcal{B}(k,m)\mathcal{B}(m,l+1)g_l^*(\xi)\big\|e^{(a-\varrho)(k-m)}\\ &\leqslant e^{(a-\varrho)(n-m)}\sum_{k\geqslant m}\big\|\mathcal{B}(k,n)\xi\big\|e^{(a-\varrho)(k-n)}\\ &\quad + \sum_{l=n}^{m-1}\sum_{k\geqslant m}\big\|\mathcal{B}(k,l+1)\big\|\cdot\big\|g_l^*(\xi)\big\|e^{(a-\varrho)(k-m)}\\ &\leqslant e^{-(a-\varrho)(m-n)}\|\xi\|_n' + \delta De^{\varepsilon+a}\|\xi\|_n'\sum_{l=n}^{m-1}\sum_{k\geqslant l}e^{\rho(l)}e^{a(l-m)-\varrho(k-m)}e^{-\varepsilon l}\\ &\leqslant e^{-(a-\varrho)(m-n)}\|\xi\|_n' + \delta De^{\varepsilon+a}\|\xi\|_n' e^{-(a-2\varrho)(m-n)}\sum_{l=n}^{m-1}e^{\varrho(l-m)}\sum_{k\geqslant l}e^{\varrho(l-k)}\\ &\leqslant e^{-(a-\varrho)(m-n)}\|\xi\|_n' + \frac{\delta De^{\varepsilon+a}}{(1-e^{-\varrho})^2}\|\xi\|_n' e^{-(a-2\varrho)(m-n)}\\ &= \big(e^{-\varrho(m-n)}+\theta\big)e^{-(a-2\varrho)(m-n)}\|\xi\|_n',\end{aligned} \tag{24}$$

where $\theta = \delta De^{\varepsilon+a}/(1-e^{-\varrho})^2$. Taking δ sufficiently small we obtain

$$\big\|(Jx)_m(\xi)\big\|_m' \leqslant \big(e^{-\varrho}+\theta\big)e^{\rho(m)}\|\xi\|_n' \leqslant e^{\rho(m)}\|\xi\|_n', \tag{25}$$

and $\|Jx\|' \leqslant 1$. Hence, $Jx \in \mathcal{B}_n$ and $J : \mathcal{B}_n \to \mathcal{B}_n$ is a well-defined operator.

We now show that J is a contraction. Given $x, y \in \mathcal{B}_n$ and $l \geqslant n$, proceeding as in (23), and setting

$$L_l = \tilde{g}_l\big(\tilde{x}_l(\xi), \tilde{\varphi}_l\big(\tilde{x}_l(\xi)\big)\big) - \tilde{g}_l\big(\tilde{y}_l(\xi), \tilde{\varphi}_l\big(\tilde{y}_l(\xi)\big)\big),$$

we obtain

$$\|L_l\| \leqslant \delta e^{-2\varepsilon l} e^{\rho(l)} \|\xi\|_n' \|x-y\|'.$$

Proceeding now as in (24) yields

$$\begin{aligned}\big\|(Jx)_m(\xi)-(Jy)_m(\xi)\big\|_m' &\leqslant \sum_{k\geqslant m}\sum_{l=n}^{m-1} \big\|\mathcal{B}(k,l+1)\big\|\cdot\|L_l\| e^{(a-\varrho)(k-m)}\\ &\leqslant \theta\|\xi\|_n' e^{\rho(m)}\|x-y\|'.\end{aligned}$$

Therefore, $\|Jx-Jy\|' \leqslant \theta\|x-y\|'$ with $\theta<1$ (see (25)), and J is a contraction. Thus, there exists a unique $x=x_\varphi \in \mathcal{B}_n$ such that $Jx=x$. This completes the proof of the lemma. □

We note that in view of (20) each function $x_{m,\varphi}$ in Lemma 1 satisfies

$$\big\|x_{m,\varphi}(\xi)\big\|_m' \leqslant e^{\rho(m)}\|\xi\|_n', \quad m\geqslant n. \tag{26}$$

3.3. Auxiliary bounds

We now obtain some information on how each function $x_{m,\varphi}$ in Lemma 2 varies with ξ.

Lemma 2. *Provided that δ is sufficiently small, for every $\varphi\in\mathcal{X}$, $n\in\mathbb{N}$, and $\xi,\bar\xi\in\Delta_n(E)$ we have*

$$\big\|\tilde{x}_{m,\varphi}(\xi)-\tilde{x}_{m,\varphi}(\bar\xi)\big\|_m' \leqslant 2\|\xi-\bar\xi\|_n' e^{\rho(m)}, \quad m\geqslant n.$$

Proof. Take $l\geqslant n$. Using the notation in (22) and proceeding as in (23), we obtain

$$\big\|g_l^*(\xi)-g_l^*(\bar\xi)\big\| \leqslant \delta e^{-2\varepsilon l}\big\|\tilde{x}_l(\xi)-\tilde{x}_l(\bar\xi)\big\|_l'.$$

Set now

$$z(l)=\big\|\tilde{x}_{l,\varphi}(\xi)-\tilde{x}_{l,\varphi}(\bar\xi)\big\|_l' \quad\text{and}\quad S(l)=e^{-\rho(l)}z(l)$$

for each $l\geqslant n$. Proceeding as in (24) and using the definition of θ, we obtain

$$\begin{aligned}z(m) \leqslant{}& \sum_{k\geqslant m}\big\|\mathcal{B}(k,n)(\xi-\bar\xi)\big\| e^{(a-\varrho)(k-m)}\\ &+\sum_{l=n}^{m-1}\sum_{k\geqslant m}\big\|\mathcal{B}(k,l+1)\big\|\cdot\big\|g_l^*(\xi)-g_l^*(\bar\xi)\big\| e^{(a-\varrho)(k-m)}\end{aligned}$$

$$\leqslant e^{-\varrho(m-n)}e^{\rho(m)}\|\xi-\bar{\xi}\|_n' + \delta D e^{a+\varepsilon}e^{\rho(m)}\sum_{l=n}^{m-1}e^{\varrho(l-m)}e^{-\rho(l)}z(l)\sum_{k\geqslant l}e^{\varrho(l-k)}$$

$$\leqslant e^{\rho(m)}\left(e^{-\varrho(m-n)}\|\xi-\bar{\xi}\|_n' + \theta\bigl(1-e^{-\varrho}\bigr)\sum_{l=n}^{m-1}e^{\varrho(l-m)}S(l)\right). \tag{27}$$

We now set $S=\sup\{S(j)\colon j\geqslant n\}$. Taking δ sufficiently small it follows from (27) that $S\leqslant \|\xi-\bar{\xi}\|_n' + S/2$, and this implies the desired statement. □

We also need some information on how each function $x_{m,\varphi}$ varies with φ. Given $\varphi,\psi\in\mathfrak{X}$ and $n\in\mathbb{N}$, let x_φ and x_ψ be the sequences given by Lemma 1.

Lemma 3. *Provided that δ is sufficiently small, for every $\varphi,\psi\in\mathfrak{X}$, $n\in\mathbb{N}$, and $\xi\in\Delta_n(E)$ we have*

$$\bigl\|\tilde{x}_{m,\varphi}(\xi)-\tilde{x}_{m,\psi}(\xi)\bigr\|_m' \leqslant \|\xi\|_n' |||\varphi-\psi||| e^{\rho(m)}, \quad m\geqslant n.$$

Proof. Take $l\geqslant n$. Proceeding as in (23) and using (18) we obtain

$$\begin{aligned}
&\bigl\|\tilde{g}_l\bigl(\tilde{x}_{l,\varphi}(\xi),\tilde{\varphi}_l\bigl(\tilde{x}_{l,\varphi}(\xi)\bigr)\bigr)-\tilde{g}_l\bigl(\tilde{x}_{l,\psi}(\xi),\tilde{\psi}_l\bigl(\tilde{x}_{l,\psi}(\xi)\bigr)\bigr)\bigr\|\\
&\quad\leqslant \delta e^{-2\varepsilon l}\bigl\|\bigl(\tilde{x}_{l,\varphi}(\xi)-\tilde{x}_{l,\psi}(\xi),\tilde{\varphi}_l\bigl(\tilde{x}_{l,\varphi}(\xi)\bigr)-\tilde{\psi}_l\bigl(\tilde{x}_{l,\psi}(\xi)\bigr)\bigr)\bigr\|\\
&\quad\leqslant \delta e^{-2\varepsilon l}\bigl(\bigl\|\tilde{x}_{l,\varphi}(\xi)\bigr\|\cdot|||\varphi-\psi|||+2\bigl\|\tilde{x}_{l,\varphi}(\xi)-\tilde{x}_{l,\psi}(\xi)\bigr\|\bigr)\\
&\quad\leqslant \delta e^{-2\varepsilon l}\bigl(\bigl\|\tilde{x}_{l,\varphi}(\xi)\bigr\|_l'|||\varphi-\psi|||+2\bigl\|\tilde{x}_{l,\varphi}(\xi)-\tilde{x}_{l,\psi}(\xi)\bigr\|_l'\bigr)\\
&\quad\leqslant \delta e^{-2\varepsilon l}e^{\rho(l)}\|\xi\|_n'|||\varphi-\psi|||+2\delta e^{-2\varepsilon l}\bigl\|\tilde{x}_{l,\varphi}(\xi)-\tilde{x}_{l,\psi}(\xi)\bigr\|_l'.
\end{aligned} \tag{28}$$

Set now

$$\bar{\rho}(l)=\bigl\|\tilde{x}_{l,\varphi}(\xi)-\tilde{x}_{l,\psi}(\xi)\bigr\|_l' \quad\text{and}\quad T(l)=e^{-\rho(l)}\bar{\rho}(l)$$

for each $l\geqslant n$. Proceeding as in (27) (see also (23) and (24)) we obtain

$$\begin{aligned}
\bar{\rho}(m)&\leqslant\sum_{l=n}^{m-1}\sum_{k\geqslant m}\bigl\|\mathcal{B}(k,l+1)\bigr\|\\
&\quad\times\bigl\|\tilde{g}_l\bigl(\tilde{x}_{l,\varphi}(\xi),\tilde{\varphi}_l\bigl(\tilde{x}_{l,\varphi}(\xi)\bigr)\bigr)-\tilde{g}_l\bigl(\tilde{x}_{l,\psi}(\xi),\tilde{\psi}_l\bigl(\tilde{x}_{l,\psi}(\xi)\bigr)\bigr)\bigr\|e^{(a-\varrho)(k-m)}\\
&\leqslant\theta e^{\rho(m)}\|\xi\|_n'|||\varphi-\psi|||+\theta\bigl(1-e^{-\varrho}\bigr)e^{\rho(m)}\sum_{l=n}^{m-1}e^{\varrho(l-m)}T(l).
\end{aligned}$$

We now set $T=\sup\{T(l)\colon l\geqslant n\}$. Taking δ sufficiently small yields

$$T\leqslant\frac{1}{2}\|\xi\|_n'|||\varphi-\psi|||+\frac{T}{2},$$

and this implies the desired statement. □

3.4. Existence of the graph

We can now use the former lemmas to establish the existence of a sequence $\varphi \in \mathcal{X}$ that later will be shown to be the desired sequence in Theorem 1.

Lemma 4. *Provided that δ is sufficiently small, there exists a unique $\varphi \in \mathcal{X}$ such that for every $n \in \mathbb{N}$ and $\xi \in B_n(E)$ we have*

$$\varphi_n(\xi) = -\sum_{l=n}^{+\infty} \mathcal{C}(l+1,n)^{-1} h_l\big(x_{l,\varphi}(\xi), \varphi_l\big(x_{l,\varphi}(\xi)\big)\big). \tag{29}$$

Proof. We look for a fixed point of the operator Φ defined for each $\varphi \in \mathcal{X}$ by

$$(\Phi\varphi)_n(\xi) = -\sum_{l=n}^{+\infty} \mathcal{C}(l+1,n)^{-1} \tilde{h}_l\big(\tilde{x}_{l,\varphi}(\xi), \tilde{\varphi}_l\big(\tilde{x}_{l,\varphi}(\xi)\big)\big) \tag{30}$$

for $n \in \mathbb{N}$ and $\xi \in B_n(E)$ where $(x_{l,\varphi})_{l \geqslant n}$ is the unique sequence given by Lemma 1. We first show that the series in (30) converges uniformly on $\Delta_n(E)$. Indeed, by (26) and (9), writing

$$h_l^*(\xi) = \tilde{h}_l\big(\tilde{x}_{l,\varphi}(\xi), \tilde{\varphi}_l\big(\tilde{x}_{l,\varphi}(\xi)\big)\big)$$

and proceeding as in (23), we obtain

$$\big\|h_l^*(\xi)\big\| \leqslant \delta e^{-2\varepsilon l} e^{\rho(l)} \|\xi\|_n' \leqslant C\delta e^{-2\varepsilon l} e^{\rho(l)} e^{\varepsilon n} \|\xi\|.$$

It follows from the second inequality in (5) that for every $p \geqslant n$,

$$\sum_{l=p}^{+\infty} \big\|\mathcal{C}(l+1,n)^{-1}\big\| \cdot \big\|h_l^*(\xi)\big\| \leqslant C\delta D \|\xi\| e^{\varepsilon - b} \sum_{l=p}^{+\infty} e^{-(T+\varepsilon)(l-n)}, \tag{31}$$

where $T = a + b - 2\varrho > 0$. Thus the series in (30) converges uniformly on $\Delta_n(E)$, and the right-hand side of (30) defines a holomorphic extension of $(\Phi\varphi)_n$ to the interior of $\Delta_n(E)$ which is continuous on $\Delta_n(E)$ (we continue to denote the extension by $(\Phi\varphi)_n$). Since $x_{l,\varphi}(0) = 0$ for every $\varphi \in \mathcal{X}$ and $l \geqslant n$ (see (26)), it follows from (30) that $(\Phi\varphi)_n(0) = 0$ for every $n \in \mathbb{N}$. Furthermore, also by (30) and since $d_0 h_l = 0$, we have $d_0(\Phi\varphi)_n = 0$ for every $n \in \mathbb{N}$.

By (26) and (9), proceeding as in (23) we obtain

$$a_l := \big\|h_l^*(\xi) - h_l^*(\bar{\xi})\big\| \leqslant 2\delta C e^{-2\varepsilon l} e^{\rho(l)} e^{\varepsilon n} \|\xi - \bar{\xi}\|.$$

In an analogous manner to that in (31) we have

$$\big\|(\Phi\varphi)_n(\xi) - (\Phi\varphi)_n(\bar{\xi})\big\| \leqslant \sum_{l=n}^{+\infty} \big\|\mathcal{C}_1(l+1,n)^{-1}\big\| a_l \leqslant K\delta \|\xi - \bar{\xi}\|$$

for some constant $K > 0$ independent of δ. Taking δ sufficiently small we have $\|(\Phi\varphi)_n\| \leqslant 1$ (see (11)) for every $n \in \mathbb{N}$. This shows that $\Phi(\mathcal{X}) \subset \mathcal{X}$ and $\Phi : \mathcal{X} \to \mathcal{X}$ is a well-defined operator.

We now show that $\Phi : \mathcal{X} \to \mathcal{X}$ is a contraction with the norm in (18). Given $\varphi, \psi \in \mathcal{X}$ and $n \in \mathbb{N}$, let x_φ and x_ψ be the unique sequences given by Lemma 1. Proceeding as in (28), and using Lemma 3, (26), and (9), we obtain

$$\begin{aligned} b_l &:= \big\| \tilde{h}_l\big(\tilde{x}_{l,\varphi}(\xi), \tilde{\varphi}_l\big(\tilde{x}_{l,\varphi}(\xi)\big)\big) - \tilde{h}_l\big(\tilde{x}_{l,\psi}(\xi), \tilde{\psi}_l\big(\tilde{x}_{l,\psi}(\xi)\big)\big) \big\| \\ &\leqslant 3C\delta e^{-2\varepsilon l} e^{\rho(l)} e^{\varepsilon n} \|\xi\| \cdot |||\varphi - \psi|||. \end{aligned}$$

In an analogous manner to that in (31) we conclude that

$$\big\|(\Phi\varphi)_n(\xi) - (\Phi\psi)_n(\xi)\big\| \leqslant \sum_{l=n}^{+\infty} \big\|\mathcal{C}(l+1,n)^{-1}\big\| b_l \leqslant \frac{3}{2} K\delta \|\xi\| \cdot |||\varphi - \psi|||.$$

Thus, taking δ sufficiently small, the operator $\Phi : \mathcal{X} \to \mathcal{X}$ is a contraction in the complete metric space $\mathcal{X}$. Hence, there exists a unique $\varphi \in \mathcal{X}$ satisfying $\Phi\varphi = \varphi$. This completes the proof of the lemma. □

We can now establish Theorem 1.

Proof of Theorem 1. Consider the unique $\varphi \in \mathcal{X}$ given by Lemma 4. Since $\mathcal{C}(m,n)\mathcal{C}(l+1,n)^{-1} = \mathcal{C}(m,l+1)$, for every $\xi \in B_n(E)$ and $m \geqslant n$ it follows from (29) that

$$\begin{aligned} &\mathcal{C}(m,n)\varphi_n(\xi) + \sum_{l=n}^{m-1} \mathcal{C}(m,l+1) h_l\big(x_l(\xi), \varphi_l\big(x_l(\xi)\big)\big) \\ &\quad = -\sum_{l=m}^{+\infty} \mathcal{C}(l+1,m)^{-1} h_l\big(x_l(\xi), \varphi_l\big(x_l(\xi)\big)\big). \end{aligned} \tag{32}$$

Given $m \geqslant n$ we set

$$\mathcal{G}(m,n) = \begin{cases} G_{m-1} \circ \cdots \circ G_n, & m > n, \\ \mathrm{Id}, & m = n, \end{cases}$$

where for each $n \in \mathbb{N}$,

$$G_n(\xi) = B_n\xi + g_n\big(\xi, \varphi_n(\xi)\big).$$

One can easily verify that the right-hand side of the first equation in (17) coincides with $\mathcal{G}(m,n)\xi$, i.e., $x_{\varphi,m}(\xi) = \mathcal{G}(m,n)\xi$. We know from (21) that $x_m(\xi) \in B_m(E)$, and hence it follows from (29) that the right-hand side of (32) is $\varphi_m(x_m(\xi))$. This establishes (15).

It remains to establish the additional properties in the theorem. By Lemma 2 and (9), for each $\xi \in \operatorname{int} B_n(E)$ we have

$$\begin{aligned} \|d_\xi x_m\| &\leqslant \sup\left\{ \frac{\|\tilde{x}_m(\xi+h) - \tilde{x}_m(\xi)\|}{\|h\|} \colon \xi, \xi+h \in \Delta_n(E),\ h \neq 0 \right\} \\ &\leqslant 2Ce^{\rho(m)+\varepsilon n}. \end{aligned} \tag{33}$$

By the mean value theorem and Lemma 2, for every $m \in \mathbb{N}$ and $\xi, \bar{\xi} \in B_m(E)$ we have

$$\begin{aligned}\|v_{mn}(\xi) - v_{mn}(\bar{\xi})\| &= \|x_m(\xi) - x_m(\bar{\xi})\| + \|\varphi_m(x_m(\xi)) - \varphi_m(x_m(\bar{\xi}))\| \\ &\leqslant 2\|x_m(\xi) - x_m(\bar{\xi})\| \leqslant 2Ce^{\rho(m)+\varepsilon n}\|\xi - \bar{\xi}\|.\end{aligned}$$

This completes the proof of the theorem. □

4. Control of derivatives along the stable manifold

We now show that all derivatives of the functions v_{mn} in (14) exhibit an exponential decay similar to that in (16), with the same exponential speed.

Theorem 2. *Assume that* (S1)–(S2) *hold. If $(A_m)_{m\in\mathbb{N}}$ admits a weak nonuniform exponential dichotomy, then provided that δ in* (7) *is sufficiently small, for the unique $\varphi \in \mathcal{X}$ in Theorem* 1 *and every $\varrho \in (0, a/2)$ there exists $\kappa > 1$ such that given $j \in \mathbb{N}$, $m \geqslant n \geqslant 0$, and $\xi, \bar{\xi} \in B_n(E)$ we have*

$$\|d_\xi^j v_{mn} - d_{\bar{\xi}}^j v_{mn}\| \leqslant \kappa^j e^{(-a+2\varrho)(m-n)+\varepsilon(j+1)n}\|\xi - \bar{\xi}\|.$$

Proof. We start with an auxiliary lemma. For each $\sigma \in (0, 1]$ we set

$$\Delta(\sigma) = \{z = (z_1, \ldots, z_k) \in \mathbb{C}^k \colon |z_i| \leqslant \sigma \text{ for } i = 1, \ldots, k\}.$$

Lemma 5. *Let $f : \operatorname{int}\Delta(1) \to \mathbb{C}$ be a holomorphic function. There exists $d = d(k) > 0$ such that for every $\rho \in (0, 1]$ and $j = 1, \ldots, k$ we have*

$$\sup_{z\in\Delta(\sigma e^{-\rho})}\left|\frac{\partial f}{\partial z_j}(z)\right| \leqslant \frac{d}{\sigma\rho^{k+1}}\sup_{z\in\Delta(\sigma)}|f(z)|. \tag{34}$$

Proof. We use the usual multi-index notation for vectors $j = (j_1, \ldots, j_k) \in \mathbb{N}_0^k$. We have $f(z) = \sum_{j\in\mathbb{N}_0^k} f_j z^j$ on the interior of $\Delta(1)$, with coefficients (see for example [5])

$$f_j = \frac{1}{(2\pi i)^k}\int_B \frac{f(z)}{z_1^{j_1+1}\cdots z_k^{j_k+1}}dz_1\cdots dz_k,$$

where

$$B = \{z = (z_1, \ldots, z_k) \in \mathbb{C}^k \colon |z_i| = \sigma \text{ for } i = 1, \ldots, k\}.$$

Therefore, setting $\Delta = \Delta(\sigma)$,

$$|f_j| \leqslant \frac{\sup_{z\in\Delta}|f(z)|}{(2\pi)^k}\int_B \frac{|dz_1|\cdots|dz_n|}{|z_1|^{j_1+1}\cdots|z_n|^{j_n+1}} \leqslant \frac{(2\pi)^k}{\sigma^{|j|}}\sup_{z\in\Delta}|f(z)|.$$

Furthermore, since $z \in \Delta(\sigma e^{-\rho})$ we obtain

$$\left|\frac{\partial f}{\partial z_j}(z)\right| \leqslant \sum_{l\in\mathbb{N}_0^k} l_j|f_l|\cdot|z|^{|l|-1} \leqslant (2\pi)^k \sup_{z\in\Delta}\left|f(z)\right| \sum_{l\in\mathbb{N}_0^k} l_j\frac{|z|^{|l|-1}}{\sigma^{|l|}}$$
$$\leqslant (2\pi)^k\frac{\sup_{z\in\Delta}|f(z)|}{\sigma}\sum_{l\in\mathbb{N}_0^k} l_j e^{-\rho(|l|-1)}.$$

We claim that for each $k, m \in \mathbb{N}$ we have

$$\operatorname{card}\left\{j\in\mathbb{N}_0^k\colon |j|=m\right\} \leqslant km^{k-1}. \tag{35}$$

Clearly (35) holds for $k=1$, and we can easily prove the claim by induction in k. Namely, if (35) holds for $k=1,\dots,l-1$, then

$$\operatorname{card}\left\{j\in\mathbb{N}_0^l\colon |j|=m\right\} \leqslant \sum_{i=0}^m \operatorname{card}\left\{j\in\mathbb{N}_0^{l-1}\colon |j|=i\right\} \leqslant (l-1)\sum_{i=1}^m i^{l-2}$$
$$\leqslant (l-1)\sum_{i=1}^m m^{l-2} = (l-1)m^{l-1} \leqslant lm^{l-1}.$$

Therefore, using (35) we obtain

$$\left|\frac{\partial f}{\partial z_j}(z)\right| \leqslant \frac{(2\pi)^k k e^{\rho}\sup_{z\in\Delta}|f(z)|}{\sigma}\sum_{m=1}^{+\infty} m^k e^{-\rho m}. \tag{36}$$

Since the maximum of $x\mapsto x^k e^{-\rho x}$ is reached only when $x=k/\rho$, we can estimate the series in (36) by this maximum times $(k/\rho)+1$ plus the integral

$$\int_0^{+\infty} x^k e^{-\rho x}\,dx = \Gamma(k+1)/\rho^{k+1},$$

that is,

$$\sum_{m=1}^{+\infty} m^k e^{-\rho m} \leqslant \left(\frac{k}{\rho e}\right)^k\left(\frac{k}{\rho}+1\right) + \frac{\Gamma(k+1)}{\rho^{k+1}} \leqslant \frac{1}{\rho^{k+1}}\left[\left(\frac{k}{e}\right)^k 2k + k!\right].$$

It follows from (36) that there exists a constant $d=d(k)>0$ for which (34) holds. This completes the proof of the lemma. □

We now proceed with the proof of the theorem. We consider a sequence $(\sigma_j)_{j\in\mathbb{N}_0}$ satisfying

$$\sigma_j = \sigma_{j-1}e^{-1/j^2} \quad \text{for } j\geqslant 1, \tag{37}$$

and we set

$$T(j)=\begin{cases}1, & j=2,\\ \prod_{l=1}^{j-2}\prod_{i=1}^{l}e^{1/i^2}, & j\geqslant 3.\end{cases}$$

We have

$$\sigma_l=\sigma_0\prod_{r=1}^{l}e^{-1/r^2} \quad\text{and}\quad \lim_{l\to\infty}\sigma_l=\sigma_0 e^{-\pi^2/6}. \tag{38}$$

Set now $\varphi_m^*(\xi)=\tilde{\varphi}_m(\tilde{x}_m(\xi))$. By the chain rule,

$$\left\|d_\xi\varphi_m^*\right\|\leqslant\|d_{x_m(\xi)}\tilde{\varphi}_m\|\cdot\|d_\xi\tilde{x}_m\|.$$

For each $\xi\in\operatorname{int}\Delta_m(E)$ we have

$$\|d_\xi\tilde{\varphi}_m\|\leqslant\sup\left\{\frac{\|\tilde{\varphi}_m(\xi+h)-\tilde{\varphi}_m(\xi)\|}{\|h\|}\colon\ \xi,\xi+h\in\Delta_m(E),\ h\neq 0\right\}\leqslant 1,$$

and thus,

$$b_1:=\sup\left\{\left\|d_\xi\tilde{\varphi}_m^*\right\|\colon\ \xi\in\Delta(\sigma_0)\right\}\leqslant\sup\left\{\|d_\xi\tilde{x}_m\|\colon\ \xi\in\Delta(\sigma_0)\right\}=:a_1. \tag{39}$$

We claim that for each $j\geqslant 2$,

$$a_j:=\sup\left\{\left\|d_\xi^j\tilde{x}_m\right\|\colon\ \xi\in\Delta(\sigma_{j-1})\right\}\leqslant\frac{T(j)}{\sigma_0^{j-1}}\prod_{l=1}^{j-1}\left(de^{(k+1)/l^2}\right)a_1 \tag{40}$$

and

$$b_j:=\sup\left\{\left\|d_\xi^j\tilde{\varphi}_m^*\right\|\colon\ \xi\in\Delta(\sigma_{j-1})\right\}\leqslant\frac{T(j)}{\sigma_0^{j-1}}\prod_{l=1}^{j-1}\left(de^{(k+1)/l^2}\right)a_1. \tag{41}$$

For $j=2$, by Lemma 5 and (39) we have

$$a_2\leqslant\frac{de^{k+1}}{\sigma_0}a_1 \quad\text{and}\quad b_2\leqslant\frac{de^{k+1}}{\sigma_0}b_1\leqslant\frac{de^{k+1}}{\sigma_0}a_1.$$

This proves (40) and (41) for $j=2$. We now assume that (40) and (41) hold for $j=l-1$ ($l\geqslant 3$). Then, by Lemma 5,

$$a_l\leqslant\frac{de^{(k+1)/(l-1)^2}}{\sigma_{l-2}}a_{l-1}\leqslant\frac{T(l-1)}{\sigma_{l-2}\sigma_0^{l-2}}\prod_{j=1}^{l-1}\left(de^{(k+1)/j^2}\right)a_1. \tag{42}$$

By (38) we have

$$\sigma_0 \frac{T(l-1)}{\sigma_{l-2}} = \prod_{j=1}^{l-3} \prod_{i=1}^{j} e^{1/i^2} \prod_{r=1}^{l-2} e^{1/r^2} = \prod_{j=1}^{l-2} \prod_{i=1}^{j} e^{1/i^2} = T(l),$$

and we conclude from (42) that

$$a_l \leqslant \frac{T(l)}{\sigma_0^{l-1}} \prod_{j=1}^{l-1} \big(d e^{(k+1)/j^2}\big) a_1,$$

i.e., (40) holds for $j = l$. Analogously, we show that (41) holds for $j = l$.

We now establish the statement in the theorem. It follows from (9) that

$$C^{-1} e^{-\varepsilon n} \Delta(E) \subset \Delta_n(E),$$

and we take $\sigma_0 = C^{-1} e^{-\varepsilon n} < 1$ in (37) (for each fixed n). Furthermore, proceeding as in (33) we obtain $a_1 \leqslant 2Ce^{\rho(m)+\varepsilon n}$. By the mean value theorem, (40), and (41), for each $j \geqslant 1$ and $\xi, \bar{\xi} \in B(\sigma_0 e^{-\pi^2/6})$ (note that this ball is contained in $\Delta(\sigma_j)$ for every j) we obtain

$$\begin{aligned}
\big\| d_\xi^j v_{mn} - d_{\bar{\xi}}^j v_{mn} \big\| &= \big\| d_\xi^j x_m - d_{\bar{\xi}}^j x_m \big\| + \big\| d_\xi^j \varphi_m^* - d_{\bar{\xi}}^j \varphi_m^* \big\| \\
&\leqslant (a_{j+1} + b_{j+1}) \|\xi - \bar{\xi}\| \\
&\leqslant 4C^{j+1} e^{\rho(m)+\varepsilon(j+1)n} T(j+1) \prod_{l=1}^{j} \big(d e^{(k+1)/l^2}\big) \|\xi - \bar{\xi}\| \\
&\leqslant 4C(dC)^j e^{(k+1)\pi^2/6} e^{\rho(m)+\varepsilon(j+1)n} T(j+1) \|\xi - \bar{\xi}\|.
\end{aligned}$$

Noticing that

$$\log T(j+1) = \sum_{l=1}^{j-1} \sum_{i=1}^{l} \frac{1}{i^2} = \sum_{i=1}^{j-1} \frac{j-i}{i^2} \leqslant j \frac{\pi^2}{6},$$

we obtain the desired statement taking $\kappa > 4dC^2 e^{(k+2)\pi^2/6}$. $\square$

5. Center manifolds

In this section we study the existence of analytic invariant center manifolds composed of trajectories $(v_m)_{m \in \mathbb{Z}}$ of (1). The approach is analogous to that in Section 2 in the case of stable manifolds, although it requires several modifications. We note that we are only able to deal with perturbations without central component.

5.1. Setup

We continue to consider the space $\mathcal{H}$ of analytic functions introduced in Section 2.1. We assume that:

(C1) there exist invertible $k \times k$ real matrices A_m, $m \in \mathbb{Z}$, such that for some invariant decomposition $\mathbb{R}^k = E \times F_1 \times F_2$ (independent of m) we have

$$A_m = \begin{pmatrix} B_m & 0 & 0 \\ 0 & C_{1m} & 0 \\ 0 & 0 & C_{2m} \end{pmatrix} \quad \text{for each } m \in \mathbb{Z}; \tag{43}$$

(C2) there exist maps $f_m \in \mathcal{H}$, $m \in \mathbb{Z}$, such that $f_m = (0, h_{1m}, h_{2m}) \in E \times F_1 \times F_2$ with $F_m = A_m + f_m$ invertible for each $m \in \mathbb{Z}$, and a constant $\delta \in (0,1)$ satisfying

$$\|f_m\| \leqslant \delta e^{-\varepsilon|m|} \quad \text{for each } m \in \mathbb{Z}. \tag{44}$$

Due to the block form in (43), each sequence $(v_m)_{m\in\mathbb{Z}} \subset \mathbb{R}^k$ satisfying $v_{m+1} = A_m v_m$ for every $m \in \mathbb{Z}$ can be written in the form

$$v_m = \big(\mathcal{B}(m,n)x_n, \mathcal{C}_1(m,n)y_{1n}, \mathcal{C}_2(m,n)y_{2n}\big) \quad \text{for every } m, n \in \mathbb{Z},$$

where $v_n = (x_n, y_{1n}, y_{2n}) \in E \times F_1 \times F_2$, and for each $m, n \in \mathbb{Z}$,

$$\mathcal{B}(m,n) = \begin{cases} B_{m-1} \cdots B_n, & m > n, \\ \mathrm{Id}, & m = n, \\ B_m^{-1} \cdots B_{n-1}^{-1}, & m < n, \end{cases}$$

with analogous definitions for $\mathcal{C}_1(m,n)$ and $\mathcal{C}_2(m,n)$. We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits *a weak nonuniform exponential trichotomy with isometric central part* if

$$\mathcal{B}(m,n) \quad \text{is an isometry for every } m, n \in \mathbb{Z},$$

and there exist constants $b > 0$, $\varepsilon \geqslant 0$, and $D \geqslant 1$ such that for every $m, n \in \mathbb{Z}$ with $m \geqslant n$ we have

$$\big\|\mathcal{C}_1(m,n)\big\| \leqslant De^{-b(m-n)+\varepsilon|n|}, \qquad \big\|\mathcal{C}_2(m,n)^{-1}\big\| \leqslant De^{-b(m-n)+\varepsilon|m|}. \tag{45}$$

5.2. Existence of center manifolds

As in Section 2.1, we denote by $B(E) \subset E$ the unit ball centered at zero, and by $\Delta(E) \subset \widetilde{E}$ the polydisk in (6). Let now $\mathcal{Y}$ be the space of sequences $(\varphi_m)_{m\in\mathbb{Z}}$ of analytic functions $\varphi_m = (\varphi_{1m}, \varphi_{2m}) : B(E) \to F_1 \times F_2$, $m \in \mathbb{Z}$, with a holomorphic extension $\tilde{\varphi}_m$ to the interior of the polydisk $\Delta(E)$ which is continuous on $\Delta(E)$, and such that for every $m \in \mathbb{Z}$, $\varphi_m(0) = 0$, $d_0\varphi_m = 0$, and

$$\|\varphi_m\| := \sup\left\{ \frac{\|\tilde{\varphi}_m(\xi) - \tilde{\varphi}_m(\bar{\xi})\|}{\|\xi - \bar{\xi}\|} \colon \xi, \bar{\xi} \in \Delta(E) \text{ with } \xi \neq \bar{\xi} \right\}.$$

Given $\varphi = (\varphi_m)_m \in \mathcal{Y}$, for each $m \in \mathbb{Z}$ we consider the graph

$$\mathcal{V}_m = \big\{ \big(\xi, \varphi_m(\xi)\big)\colon \xi \in B(E) \big\} \subset \mathbb{R}^k,$$

and given $\xi \in B(E)$ and $m, n \in \mathbb{Z}$ we set $v_{mn}(\xi) = \mathcal{F}(m,n)(\xi, \varphi_n(\xi))$, where

$$\mathcal{F}(m,n) = \begin{cases} F_{m-1} \circ \cdots \circ F_n, & m > n, \\ \mathrm{Id}, & m = n, \\ F_m^{-1} \circ \cdots \circ F_{n-1}^{-1}, & m < n. \end{cases}$$

We now present our center manifold theorem.

Theorem 3. *Assume that* (C1)–(C2) *hold. If* $(A_m)_{m\in\mathbb{Z}}$ *admits a weak nonuniform exponential trichotomy with isometric central part, then provided that* δ *in* (44) *is sufficiently small there exists a unique* $\varphi \in \mathcal{Y}$ *such that*

$$\mathcal{F}(n,m)(\mathcal{V}_m) = \mathcal{V}_n \quad \textit{for every } m, n \in \mathbb{Z}. \tag{46}$$

In addition:

1. $\mathcal{V}_m$ *is an analytic manifold,* $0 \in \mathcal{V}_m$*, and* $T_0\mathcal{V}_m = E$ *for every* $m \in \mathbb{Z}$;
2. *there exists* $\kappa > 1$ *such that for* $j \in \mathbb{N}$, $m, n \in \mathbb{Z}$, *and* $\xi, \bar{\xi} \in E$ *we have*

$$\big\| d_\xi^j v_{mn} - d_{\bar{\xi}}^j v_{mn} \big\| \leqslant \kappa^j \|\xi - \bar{\xi}\|.$$

We call each manifold $\mathcal{V}_m$ in Theorem 3 a *local center manifold*.

We note that, under the hypotheses of Theorem 3, an immediate consequence of Theorem 1 is that there also exist families of stable and unstable manifolds. For this it is enough to rewrite the block form in (43) as in (4); for example, to obtain the stable manifolds we consider the stable and center-unstable blocks, respectively given by C_{1m} and $B_m \oplus C_{2m}$.

5.3. Proof of Theorem 3

In view of the required invariance in (46), any trajectory starting in $\mathcal{V}_n$ must be in $\mathcal{V}_m$ for every $m \in \mathbb{Z}$. Thus, given $(n, \xi) \in \mathbb{Z} \times B(E)$ with $v_n = (\xi, \varphi_n(\xi)) \in \mathcal{V}_n$, and setting $x_m(\xi) = \mathcal{B}(m,n)\xi$ for each $m \in \mathbb{Z}$, the equations in (1) can be written in the form

$$\varphi_{im}\big(x_m(\xi)\big) = \mathcal{C}_i(m,n)\varphi_{in}(\xi) + \sum_{l=n}^{m-1} \mathcal{C}_i(m, l+1) h_{il}\big(x_l(\xi), \varphi_l\big(x_l(\xi)\big)\big), \quad i = 1, 2,$$

for $m \geqslant n$, and

$$\varphi_{im}\big(x_m(\xi)\big) = \mathcal{C}_i(m,n)\varphi_{in}(\xi) - \sum_{l=m}^{n-1} \mathcal{C}_i(m, l+1) h_{il}\big(x_l(\xi), \varphi_l\big(x_l(\xi)\big)\big), \quad i = 1, 2,$$

for $m \leqslant n$. The norm in $\mathbb{R}^k$ is given by

$$\|(x, y, z)\| = \max\{\|x\|, \|y\|, \|z\|\}$$

for each $(x, y, z) \in E \times F_1 \times F_2$, and we equip the space $\mathcal{Y}$ with the norm

$$|||\varphi||| = \sup\{\|\varphi_m(x)\|/\|x\|: m \in \mathbb{Z} \text{ and } x \in \Delta(E) \setminus \{0\}\}. \tag{47}$$

One can easily verify that $\mathcal{Y}$ is becomes a complete metric space.

We first establish an auxiliary statement.

Lemma 6. *Provided that δ is sufficiently small, there exists a unique $\varphi \in \mathcal{Y}$ such that for every $(n, \xi) \in \mathbb{Z} \times B(E)$ and $i = 1, 2$ we have*

$$\varphi_{in}(\xi) = a_i \sum_{l \in A_i^n} \mathcal{C}_i(l+1, n)^{-1} h_{il}(x_l(\xi), \varphi_l(x_l(\xi))), \tag{48}$$

where $a_1 = 1$, $A_1^n = (-\infty, n-1]$ and $a_2 = -1$, $A_2^n = [n, +\infty)$.

Proof. Set

$$h_{il}^*(\xi) := \tilde{h}_{il}(x_l(\xi), \tilde{\varphi}_l(x_l(\xi))).$$

We look for a fixed point of the operator Φ defined for each $\varphi \in \mathcal{Y}$ by

$$(\Phi\varphi)_n(\xi) = \left(\sum_{l=-\infty}^{n-1} \mathcal{C}_1(l+1, n)^{-1} h_{1l}^*(\xi), -\sum_{l=n}^{+\infty} \mathcal{C}_2(l+1, n)^{-1} h_{2l}^*(\xi)\right) \tag{49}$$

for $(n, \xi) \in \mathbb{Z} \times B(E)$. We first prove that each series in (49) converges uniformly on $\Delta(E)$. It follows from (45) that $\|x_l(\xi)\| = \|\xi\|$ for every $\xi \in \widetilde{E}$. Furthermore,

$$(x_l(\xi), \tilde{\varphi}_l(x_l(\xi))) \in \Delta(\mathbb{R}^n) \quad \text{for every } \xi \in \widetilde{E}.$$

Hence, we can compute $h_{il}^*(\xi)$, and

$$\|h_{il}^*(\xi)\| \leqslant \delta e^{-\varepsilon|l|}\|\xi\|, \quad i = 1, 2. \tag{50}$$

It follows from the inequalities in (45), (50), and $|l| \leqslant |l-n| + |n|$ that

$$\sum_{l \in A_i^n \setminus [-p, p]} \|\mathcal{C}_i(l+1, n)^{-1}\| \cdot \|h_{il}^*(\xi)\| \leqslant \delta D\|\xi\| e^{\varepsilon} \sum_{l \in A_i^n \setminus [-p, p]} e^{-b|n-l-1|}$$

for each $p \in \mathbb{N}$, and thus the two series in (49) converge uniformly on $\Delta(E)$. This shows that $(\Phi\varphi)_n$ is a well-defined analytic function with a holomorphic extension to the interior of $\Delta(E)$ which is continuous on $\Delta(E)$. Moreover, since $x_m(0) = 0$ for every $\varphi \in \mathcal{Y}$ and $m \in \mathbb{Z}$, it follows from (49) that $(\Phi\varphi)_n(0) = 0$ for every $n \in \mathbb{Z}$. Also by (49), and since $d_0 h_{1l} = d_0 h_{2l} = 0$, we have $d_0(\Phi\varphi)_n = 0$ for every $n \in \mathbb{Z}$.

Observe now that since

$$\left\|x_l(\xi)-x_l(\bar{\xi})\right\|=\left\|\mathcal{B}(m,n)(\xi-\bar{\xi})\right\|=\|\xi-\bar{\xi}\|, \tag{51}$$

for $i=1,2$ we have

$$a_{il}:=\left\|h^*_{il}(\xi)-h^*_{il}(\bar{\xi})\right\|\leqslant \delta e^{-\varepsilon|l|}\|\xi-\bar{\xi}\|.$$

Thus, by (45), the norm of $(\Phi\varphi)_n(\xi)-(\Phi\varphi)_n(\bar{\xi})$ is bounded by

$$\begin{aligned}&\sum_{l\in A_1^n}\left\|\mathcal{C}_1(l+1,n)^{-1}\right\|a_{2l}+\sum_{l\in A_2^n}\left\|\mathcal{C}_2(l+1,n)^{-1}\right\|a_{1l}\\&\quad\leqslant D\delta\|\xi-\bar{\xi}\|e^{\varepsilon}\left(e^b\sum_{l\in A_1^n}e^{-b(n-l)}+e^{-b}\sum_{l\in A_2^n}e^{-b(l-n)}\right)\leqslant \delta K\|\xi-\bar{\xi}\|\end{aligned}$$

for some constant $K>0$ independent of δ. Taking δ sufficiently small so that $\delta K<1$ we obtain

$$\left\|(\Phi\varphi)_n(\xi)-(\Phi\varphi)_n(\bar{\xi})\right\|\leqslant\|\xi-\bar{\xi}\|$$

for every $n\in\mathbb{Z}$ and $\xi,\bar{\xi}\in\Delta(E)$. This shows that $\Phi(\mathcal{Y})\subset\mathcal{Y}$ and the operator $\Phi:\mathcal{Y}\to\mathcal{Y}$ is well defined.

We now show that $\Phi:\mathcal{Y}\to\mathcal{Y}$ is a contraction with the norm in (47). Given $\varphi,\psi\in\mathcal{Y}$ and $(n,\xi)\in\mathbb{Z}\times\Delta(E)$, for $j=1,2$ we have

$$\begin{aligned}b_{jl}&:=\left\|\tilde{h}_{jl}\big(x_l(\xi),\tilde{\varphi}_l\big(x_l(\xi)\big)\big)-\tilde{h}_{jl}\big(x_l(\xi),\tilde{\psi}_l\big(x_l(\xi)\big)\big)\right\|\\&\leqslant\delta e^{-\varepsilon|l|}\|\xi\|\cdot|||\varphi-\psi|||.\end{aligned}$$

Using the inequalities in (45) we conclude that

$$\begin{aligned}\left\|(\Phi\varphi)_n(\xi)-(\Phi\psi)_n(\xi)\right\|&\leqslant\sum_{l\in A_1^n}\left\|\mathcal{C}_1(l+1,n)^{-1}\right\|b_{2l}+\sum_{l\in A_2^n}\left\|\mathcal{C}_2(l+1,n)^{-1}\right\|b_{1l}\\&\leqslant\delta K\|\xi\|\cdot|||\varphi-\psi|||.\end{aligned}$$

Therefore

$$|||\Phi\varphi_1-\Phi\varphi_2|||\leqslant\delta K|||\varphi_1-\varphi_2|||,$$

and $\Phi:\mathcal{Y}\to\mathcal{Y}$ is a contraction in the complete metric space $\mathcal{Y}$. Hence, there exists a unique $\varphi\in\mathcal{Y}$ satisfying $\Phi\varphi=\varphi$. This completes the proof of the lemma. □

We can now establish Theorem 3.

Proof of Theorem 3. Consider the unique $\varphi\in\mathcal{Y}$ given by Lemma 6. Since

$$\mathcal{C}_i(m,n)\mathcal{C}_i(l+1,n)^{-1}=\mathcal{C}_i(m,l+1)$$

for $i = 1, 2$, for each $\xi \in \Delta(E)$ we have

$$\begin{aligned}
&\mathcal{C}_i(m,n)\varphi_{in}(\xi) + \sum_{l=n}^{m-1} \mathcal{C}_i(m,l+1)h_{il}\big(x_l(\xi), \varphi_l\big(x_l(\xi)\big)\big) \\
&\quad = a_i \sum_{l \in A_i^m} \mathcal{C}_i(l+1,m)^{-1} h_{il}\big(x_l(\xi), \varphi_l\big(x_l(\xi)\big)\big) \\
&\quad = a_i \sum_{l \in A_i^m} \mathcal{C}_i(l+1,m)^{-1} h_{il}\big(\mathcal{B}(l,m)x_m(\xi), \varphi_l\big(\mathcal{B}(l,m)x_m(\xi)\big)\big) \qquad (52)
\end{aligned}$$

for $m \geqslant n$, and

$$\begin{aligned}
&\mathcal{C}_i(m,n)\varphi_{in}(\xi) - \sum_{l=m}^{n-1} \mathcal{C}_i(m,l+1)h_{il}\big(x_l(\xi), \varphi_l\big(x_l(\xi)\big)\big) \\
&\quad = a_i \sum_{l \in A_i^m} \mathcal{C}_i(l+1,m)^{-1} h_{il}\big(x_l(\xi), \varphi_l\big(x_l(\xi)\big)\big) \\
&\quad = a_i \sum_{l \in A_i^m} \mathcal{C}_i(l+1,m)^{-1} h_{il}\big(\mathcal{B}(l,m)x_m(\xi), \varphi_l\big(\mathcal{B}(l,m)x_m(\xi)\big)\big) \qquad (53)
\end{aligned}$$

for $m \leqslant n$. It follows from (48) that the right-hand sides of (52) and (53) are $\varphi_{im}(x_m(\xi))$. This establishes (46) and the first property in the theorem.

To obtain the second property, we set $\varphi_m^*(\xi) = \varphi_m(x_m(\xi))$ and we note that (see (51))

$$\begin{aligned}
\big\|v_{mn}(\xi) - v_{mn}(\bar{\xi})\big\| &= \big\|x_m(\xi) - x_m(\bar{\xi})\big\| + \big\|\varphi_m^*(\xi) - \varphi_m^*(\bar{\xi})\big\| \\
&\leqslant 2\big\|x_m(\xi) - x_m(\bar{\xi})\big\| = 2\|\xi - \bar{\xi}\|.
\end{aligned}$$

To obtain the estimates for the derivatives we proceed in a similar manner to that in the proof of Theorem 2. We also use the same notation. We set $\sigma_0 = 1$ in (37), and we have $a_1 = 1$ (see (39)). By the mean value theorem and (41), for each $j \geqslant 1$ we have

$$\begin{aligned}
\big\|d_\xi^j v_{mn} - d_{\bar{\xi}}^j v_{mn}\big\| &= \big\|d_\xi^j \varphi_m^* - d_{\bar{\xi}}^j \varphi_m^*\big\| \leqslant b_{j+1}\|\xi - \bar{\xi}\| \\
&\leqslant T(j+1)\prod_{l=1}^{j}\big(de^{(k+1)/l^2}\big)\|\xi - \bar{\xi}\|.
\end{aligned}$$

This completes the proof of the theorem. □

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